Cohesive Dynamics for Quasibrittle Fracture

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Classic Theory of Dynamic Fracture Mechanics

- The theory of dynamic fracture is based on the notion of a deformable continuum containing a crack.
- The crack is mathematically modeled as a branch cut that begins to move when an infinitesimal extension of the crack releases more energy than needed to create a fracture surface.
- Fracture mechanics, together with experiment, has been enormously successful in characterizing and measuring the resistance of materials to crack growth and thereby enabling engineering design.

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   - Write the discrete problem in atomic level, modeling the atomic bonds as they break.
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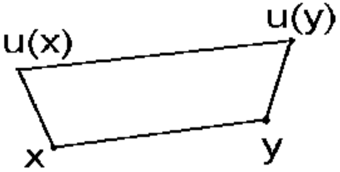
3. **Non-Local Models for fracture energies - Bottom Up Approach:** Modeling discreteness of fracture at atomistic length scales through non-local models & upscaling to classic fracture energy.

Mdeling Cracks

1. **Top Down Approach:**
   - **Cohesive Zone:** *Xu and Needleman, JMPS 42 (1994)*
   - **XFEM Implementation Of The Cohesive Zone:** *Belytscho and Black, Int.J.Numer.Meth.Eng. 45 (1995)*
   - **Phase Fields:**
     1. *Bourdin, Larsen and Richardson, Int.J.Fract. 168 (2011)*
     2. *Borden, Verhoosel, Scott, Hughes and Landis, CMAME 217 (2012)*
     3. *Miehe, Hofacker and Welschinger, CMAME 199 (2010)*
2. **Lattice Models - Bottom Up Approach:**
   - *Marder and Gross, JMPS 43 (1995)*
   - *Buehler, Abraham and Gao, Nature 426 (2005)*
3. **Non-Local Models for fracture energies - Bottom Up Approach:**
   - *Truskinovsky (1996)*

Cohesive Dynamic Modeling - Formulation

- Define the displacement function $u(t, x)$.



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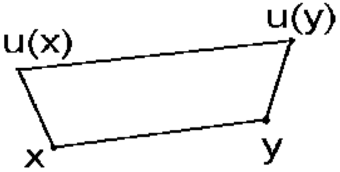

- Define the strain between two points x and y to be

$$S(y,x) = \frac{|u(y) - u(x)| - |y - x|}{|y - x|}$$

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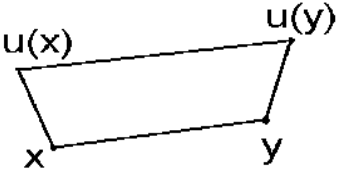

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$$S(y,x) = \frac{|u(y) - u(x)| - |y - x|}{|y - x|}$$

- Now linearize the strain for the small deformation case

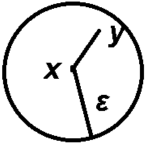
$$S(y,x) = \frac{u(y) - u(x)}{|y - x|} . \frac{y - x}{|y - x|} = \frac{u(y) - u(x)}{|y - x|} . e$$

Cohesive Dynamic Modeling - Formulation

- Adopt the non-local non-linear formulation by S.A.Silling
- Define the force on x due to the strain S between x and y to be $k^\epsilon(S, x-y)$
- The point x interacts with points y inside a neighborhood of radius ϵ.
- Summing over all points y within the neighborhood of interaction of radius ϵ to get the total force on x: $\int_{H_\epsilon(x)} k^\epsilon(S, x-y)dy$
- Apply Newton's law to get the equation of the dynamics for the displacement

$$\rho u_{tt} = \int_{H_\epsilon(x)} k^\epsilon(S, x-y)dy + b$$

Where $b(x)$ is the body force at x

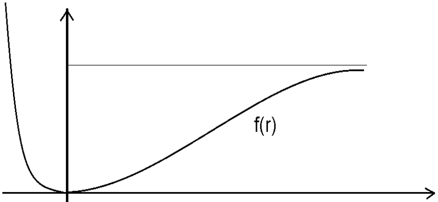


Cohesive Dynamic Modeling - Formulation

Define the peridynamic potential energy

$$PD^{\epsilon}(u) = \frac{1}{V_d}\int_D \int_{H_\epsilon(x)} |y-x|\partial_S W^{\epsilon}(|y-x|, S(u))dydx$$

- Where V_d is the volume of the neighborhood.
- $W^{\epsilon}(|y-x|, S(u)) = \frac{1}{\epsilon}J(\frac{|y-x|}{\epsilon})\frac{1}{|y-x|}f(\sqrt{|y-x|}S(u))$ is the potential energy per unit length.
- Motivated by the Lennard-Jones potential, the function $f(\sqrt{|y-x|}S)$ is the strain potential associated with the pair x and y. Here the function $f(r)$ is shown below:



Cohesive Dynamic Modeling

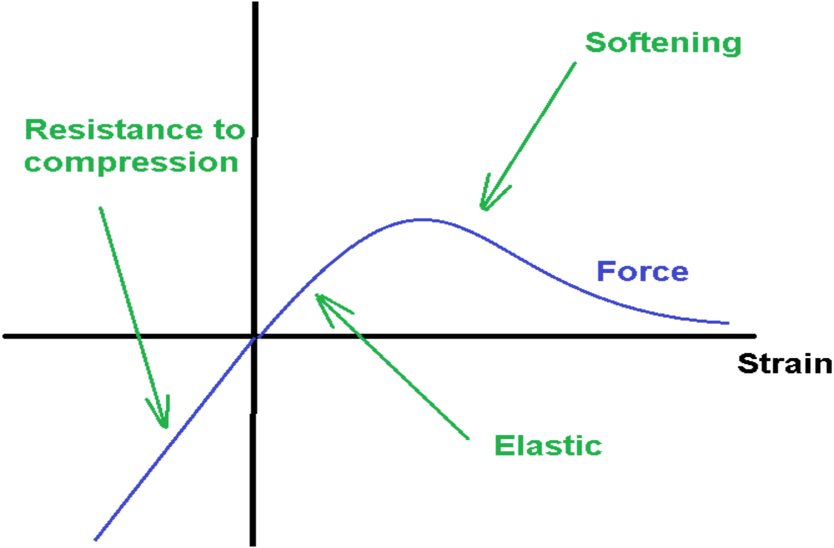
Use the Lagrangian along with the Principle of Least Action to get the model:

$$\rho u_{tt} = \frac{2}{V_d} \int_{H_\epsilon(x)} \partial_S W^\epsilon(|y-x|, S(u)) \frac{y-x}{|y-x|} dy + b$$

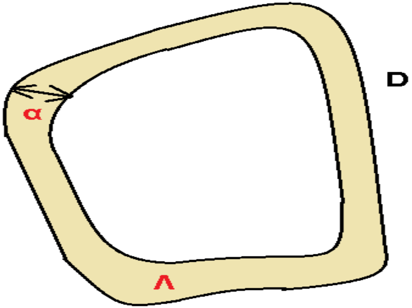
Where $\partial_S W^\epsilon(|y-x|, S(u))$ is the force on x due to the strain S between x and y

$$\partial_S W^\epsilon(|y-x|, S(u)) = \frac{1}{\epsilon} J(\frac{|y-x|}{\epsilon}) \frac{1}{\sqrt{|y-x|}} f'(\sqrt{|y-x|} S(u))$$

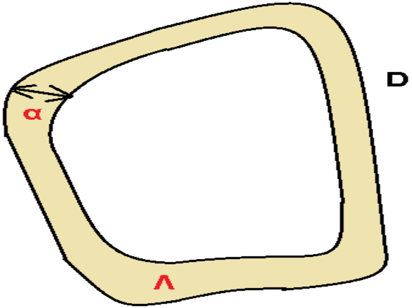
Cohesive Dynamic Modeling: Force vs Strain



Non-local Boundary Conditions



- Λ is a boundary layer of thickness α.



- Λ is a boundary layer of thickness α.
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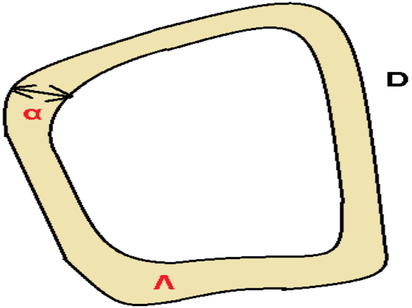

- Λ is a boundary layer of thickness α.
- Here the non-local boundary conditions for the displacement at x in Λ is $u(x) = 0$.
- Where u is in the space of functions defined on D and $\int_D |u|^2 dy < \infty$.

Initial boundary value problem and The Existence of A Solution

- Given the initial data

$$
\begin{aligned}
u(0,x) &= u_0(x) \in L^2(D,\mathbb{R}^3)\\
u_t(0,x) &= v_0(x) \in L^2(D,\mathbb{R}^3)
\end{aligned}
$$

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- It can be shown that the right hand side of the model

$$\rho u_{tt} = \frac{2}{V_d}\int_{H_\epsilon(x)} \partial_S W^\epsilon(|y-x|, S(u))\frac{y-x}{|y-x|}dy + b$$

is Lipschitz continuous in u, so there is a unique solution

$$u(t,x) \in C^2([0,T], L^2(D,\mathbb{R}^3))$$

of the initial value problem

Energy Balance Property Of The Evolution

The total energy at time t is:

$$E^{\epsilon}(t, u(t))) = \frac{\rho}{2}||u_t(t)||^2 + PD^{\epsilon}(u) - \int_D b(t).u(t)dx$$

The total energy at time $t = 0$

$$E^{\epsilon}(0, u_0) = \frac{\rho}{2}||v_0||^2 + PD^{\epsilon}(u_0) - \int_D b(0).u_0 dx$$

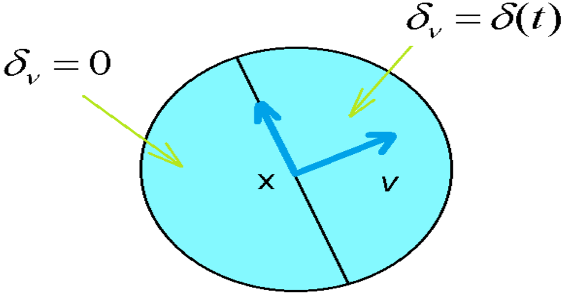
Energy balance at time t

$$E^{\epsilon}(t, u(t)) = E^{\epsilon}(0, u_0) - \int_0^t \int_D b_t(\tau).u(\tau)dxd\tau$$

Instability And Growth Of Jump Discontinuities

We wil show that dynamic instability gives rise to growth of jump discontinuity.
We start with an equilibrium solution u and perturb it with a small jump δ

$$\begin{aligned} u^p &= u + \delta \\ \delta &= \delta(x,t) = \begin{Bmatrix} 0; (y-x).\nu \leq 0 \\ \overline{u}\delta(t); (y-x).\nu > 0 \end{Bmatrix} \end{aligned}$$



Instability And Growth Of Jump Discontinuities

An application of Taylor's theorem to the difference

$$-\frac{2}{V_d}\int_{H_\epsilon(x)}[\partial_S W^\epsilon(u^p)-\partial_S W^\epsilon(u)]e dy$$

And substitution into the equation of motion delivers the dynamics for the jump discontinuity

$$\rho\overline{u}\delta_{tt}=A_\nu\overline{u}\delta(t)$$

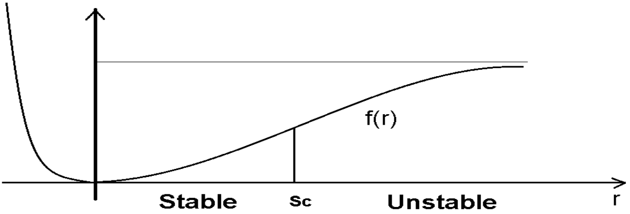
Where A_ν is the symmetric stability matrix

$$A_\nu=\frac{-2}{V_d}\int_{H_\epsilon(x)\cap E_\nu^-}\frac{1}{\epsilon}J(\frac{|y-x|}{\epsilon})f''(\sqrt{|y-x|}S)e\otimes e dy$$

Instability And Growth Of Jump Discontinuities

Now, given $A_\nu = \frac{-2}{V_d} \int_{H_\epsilon(x)\cap E_\nu^-} \frac{1}{\epsilon} J(\frac{|y-x|}{\epsilon}) f''(\sqrt{|y-x|}S) e \otimes e dy$, observe that stability vs instability follows from:

$$f'' > 0 \text{ for } S < \frac{\bar{r}}{\sqrt{\epsilon|\xi|}} = S_c$$
$$f'' < 0 \text{ for } S > \frac{\bar{r}}{\sqrt{\epsilon|\xi|}} = S_c$$



We find that a jump discontinuity is unstable if a sufficient number of bonds are stretched beyond the critical value of S_c.

Instability And Growth Of Jump Discontinuities

From *Stability Theory we get:*

- If an eigenvalue of A_v is negative then the perturbation grows and

$$\partial_S^2 W^\epsilon(S) > 0 \text{ or } S > S_c$$

Which gives an unstable solution

Instability And Growth Of Jump Discontinuities

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- If all eigenvalues of A_v are positive then the perturbation is stable and

$$\partial_S^2 W^\epsilon(S) < 0 \text{ or } S < S_c$$

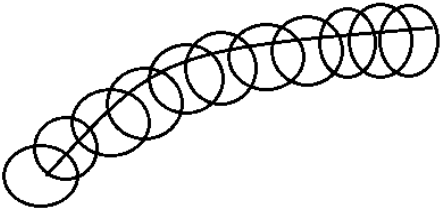
  Which gives a stable solution

Definition Of The Process Zone

Consider a point x with its neighborhood $H_\epsilon(x)$, the volume fraction of points y inside $H_\epsilon(x)$ for which $S(y,x) > S_c$ is given by

$$P(\{y \in H_\epsilon(x) : S(y,x) > S_c\})$$

We call the process zone PZ^ϵ the union of all centers x of neighborhoods for which $P(\{y \in H_\epsilon(x) : S(y,x) > S_c\}) > \Theta$, where $0 < \Theta < 1$



Properties Of The Process Zone

- From its definition, we can see that it contains jump discontinuities of the displacement field.
- From stability theory, we can see that the process zone contains points inside the body where small fissures can grow to macroscopic size.

Control Of The Process Zone - The Peridynamics Horizon As A Modeling Parameter

The volume of the process zone is controlled by ϵ according to the following fundamental inequality derived directly from the equation of motion

$$\Theta f(\overline{r}) VOL(PZ^{\epsilon}) \leq \frac{\epsilon}{m} k$$

- $VOL(PZ^{\epsilon})$ is the volume of the process zone.
- $f(\overline{r})$ is the energy per unit length needed to soften the bond.
- k is the total energy put into the system up to time T.
- Θ is the fixed proportion of bonds softened inside the horizon of x.
- m is a normalizing factor.
- ϵ is the ratio of non-local interactions to sample size.

Control Of The Process Zone - The Peridynamics Horizon As A Modeling Parameter

From the inequality

$$\Theta f(\bar{r}) VOL(PZ^{\epsilon}) \leq \frac{\epsilon}{m} k$$

We see that the volume of the process zone goes to zero with the diameter of the neighborhood ϵ

$$VOL(PZ^{\epsilon}) \leq \frac{1}{\Theta f(\bar{r})} \frac{\epsilon}{m} k$$

Hence, we can choose the peridynamic horizon ϵ based on experimental observation of the size of the process zone.
Or, alternatively, given the total energy put into the system, the size of the horizon, the force needed to soften a bond and the estimated proportion of soften bonds, we can have an estimated upper bound of the volume of the process zone.

Conclusion

- Defined a new peridynamic model based on the Lennard-Jones potential.
- Showed well posedness and the existence of a unique solution.
- Demonstrated energy balance for the evolution.
- Showed that small fissures can grow to big cracks in the process zone.
- Showed that the volume of the process zone is controled by the horizon.

THANK YOU!